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An Isomorphism for Digital Images

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In usual topology, a homeomorphism is a one to one mapping between two topological spaces which induces a one to one mapping between their open subsets and so establishes an equivalence between their topologies. In digital images, as well as in several discrete structures (e.g., planar graphs), one encounters concepts and features analogous to those of topology, for example connectedness, holes, surrounding relations, but it is impossible to define on these structures an isomorphism in the classical sense, if one excepts a trivial one, and this only between images having the same number of points for each colour. It is thus necessary to define in a new way a corresponding concept for digital images. In this paper, an isomorphism between two digital images as a relation, not a map, which satisfies several requirements related to the equivalence of the two digital structures is defined. Such an isomorphism wil then play the same role as the homeomorphism in classical topology. The requirements for this isomorphism are found by a study of the special case of binary images on a rectangular grid, on which we can construct such an isomorphism from a Euclidean plane homeomorphism thanks to a correspondence that we establish between the digital rectangular grid structure and the Euclidean plane topology. It is shown then how this new type of isomorphism preserves certain digital features related to topology (connected components, surrounding relations, etc.). These properties, together with the correspondence with the Euclidean topology in the case of the rectangular grid, validate our definition of the digital isomorphism. © 1985 Academic Press, Inc.

1. INTRODUCTION

In topology, in particular in the case of the real plane, one defines a homeomorphism as a bijection between two spaces, which induces a bijection between the sets of open subsets of these two spaces [1].

If one turns to discrete or combinatorial structures, one would like to find an equivalence between similar structures corresponding to the homeomorphism in classical topology. However, if we try to construct such a correspondence by a bijection between two sets which induces a bijection between their structures, then we will only get trivial isomorphisms, as we explain below.

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Consider the two-dimensional digital pictures used today in computer science for the representation and processing of real images. They consist of small points called picture elements or simply *pixels*, aligned on a rectangular array; each pixel corresponds physically to a small square in a square tessellation of the rectangular image and has a particular colour attached to it. This tessellation forms what we call a *rectangular grid*. See, for example, Fig. 1, where we show two digital pictures having two colours, black and white (the pixels should be seen touching each other).

We can consider two different structures in order to describe the properties of such a picture. The first one is topological in the usual sense: we identify the pixels of the grid to the squares to which they correspond in the Euclidean plane, obtaining thus the classical topological rectangle. Pixels have various colours, and the *topological image* is a partition of the rectangle into unions of pixels corresponding to each colour (but we have to decide precisely about the colour of the points along the border between two pixels having distinct colours). The second structure is combinatorial: the digital rectangular grid can be seen as a rectangular array of vertices, and we have two possible neighbourhood relations between pixels, called the 4-adjacency and the 8-adjacency. We illustrate them on Fig. 2 by representing the pixels adjacent to a given pixel p. Thus the grid becomes

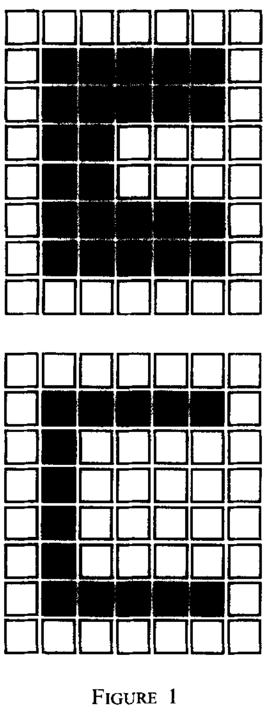
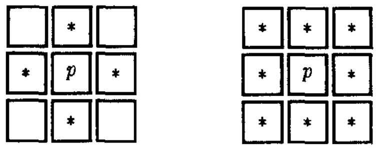


FIGURE 1



4-adjacency 8-adjacency

FIGURE 2

identified with a double graph, whose vertices are pixels and whose edges link adjacent pixels. These neighbourhood relations allow us to define on a rectangular grid concepts like connectedness, holes, etc., which are derived from topology. This neighbourhood structure is thus generally considered in the picture processing community as the digital correspondent of the usual topology, and it is therefore called the "digital topology." Now a *digital image* is a partition of the set of vertices of this double graph into subsets corresponding to each coulour.

While it is possible to find homeomorphic images on the grid if we consider their topological structure (where pixels represent Euclidean squares), it is generally impossible to define an isomorphism between the corresponding digital images through a bijection between their sets of vertices inducing a bijection between their sets of edges. Assuming that the grid is square (i.e., has the same number of rows and columns) such isomorphisms can only be geometrical symmetries of the square: rotations of a multiple of 90° and diagonal or median symmetries.

However, a wider definition of isomorphisms in digital images is needed. Consider for example the two *binary* (i.e., two-tone) rectangular grid *digital* images of Fig. 1, where the second one is in fact a thinned version of the first. As they do not have the same number of black pixels, there is no colour-preserving bijection establishing an isomorphism between them. But our intuition tells us that they have the same "topological" structure. In fact, the corresponding *topological* images (where each pixel is seen as the surface that if fills in the Euclidean plane) are homeomorphic in the usual sense.

Such a type of structural equivalence between digital images, which is not an isomorphism in the classical sense (a bijection between the sets of vertices inducing a bijection between the sets of edges), appears clearly in what one calls in digital picture processing *thinning algorithms* (see Sect. 9.4.5 of [3]): applied to a binary (black on white) digital picture, they transform it into a one-pixel thick equivalent, having a similar shape and the same "topology" with regards to connected components and holes. These algorithms are useful, for example, in character recognition, because they reduce the amount of information while keeping the main features of the image.

Our general problem is to relate properties of the two kinds of images. The purpose of this paper is to define a new type of isomorphism for digital structures of arbitrary type, in particular for digital images on a two-dimensional rectangular grid, having a given number n of colours and a structure determined by a certain number of adjacency relations. It will be such that the two images of Fig. 1 are isomorphic. Such an isomorphism will not be a 1 to 1 mapping, but a relation between the points of two images.

Let us give here the definition of a digital homeomorphism in the case of an n-coloured image on a rectangular grid:

DEFINITION. Let G and G' be two rectangular grids on which we define the two n-coloured images $\mathscr{I}$ and $\mathscr{I}'$, respectively. Let FG be the frame of G, that is the set of pixels of G along the border of the grid, and let FG' be the frame of G'. Let ϕ be a relation between G and G'. For every $X \subseteq G$ and $X' \subseteq G'$, let us define

$$X\rho = \{p' \in G' \mid \exists p \in X \text{ such that } p\phi p'\}$$

and

$$X'\lambda = \{p \in G \mid \exists p' \in X' \text{ such that } p\phi p'\}.$$

Then ϕ is a digital isomorphism between the two rectangular grid images $(G, \mathscr{I})$ and $(G', \mathscr{I}')$ if and only if it satisfies the following four conditions:

(1°) *Totality.* For $p \in G$ and $p' \in G'$, $p\rho \neq \varnothing \neq p'\lambda$.

(2°) *Frame Preservation.* For $p \in FG$ and $p' \in FG'$, $p\rho \cap FG' \neq \varnothing \neq p'\lambda \cap FG$.

(3°) *Image Preservation.* For $p \in G$ and $p' \in G'$, if $p\phi p'$, then p and p' have the same colour.

(4°) *Adjacency Preservation.* For $X \subseteq G$, $X' \subseteq G'$ and $k = 4$ or 8, we have: If X is k-connected, then $X\rho$ is k-connected; if X' is k-connected, then $X'\lambda$ is k-connected.

These requirements (1°) to (4°) for a digital isomorphism will be found through the correspondence that exists between the neighbourhood structure of a rectangular grid and the Euclidean plane topology of the the corresponding rectangular topological image formed by the small squares filled by the pixels. Indeed, a homeomorphism ψ between the two topological images corresponding to $(G, \mathscr{I})$ and $(G', \mathscr{I}')$ induces a digital isomorphism $\theta(\psi)$ between G and G'. We conjecture also that for binary rectangular grid images, every digital isomorphism ϕ can in fact be derived from a Euclidean homeomorphism ξ.

The paper is organized as follows: In Section 2 we establish the links between the structure of images on a rectangular grid embedded in the Euclidean plane and the topology of that plane. Then from a Euclidean homeomorphism ψ we can construct a relation $\theta(\psi)$ between rectangular grids which we call the *derived binary rectangular grid isomorphism*. We describe four properties of such a derived isomorphism (which are consequences of the properties of a Euclidean plane homeomorphism). We decide finally to take these four properties as the definition for a *digital isomorphism* between two binary rectangular grid images, not necessarily derived from a Euclidean plane homeomorphism.

In Section 3 we give a formal combinatorial definition of digital images in a very general way, and we translate in these terms the requirements found in Section 2 in the particular case of binary rectangular grid images. This leads to our general definition of the *digital isomorphism*. We give several examples of this new type of isomorphism.

In Section 4 we examine the properties of digital isomorphisms, for example, the preservation of connected components and of surrounding relations.

The similarity of these properties with those of the Euclidean plane homeomorphism, and the correspondence between the digital isomorphism and the Euclidean plane homeomorphism in the case of binary images on a rectangular grid, justify our definition of the digital isomorphism. Moreover, it can have practical applications. It is possible to use it for the purpose of checking the mathematical validity of several operations on digital images that are intended to be "topology-preserving": for example, the thinning algorithms described above, but also what one calls *shrinking* and *expansion*, two operations respecting the connectivity properties and holes of digital images, but discarding their geometrical shape (see Sects. 9.2.4 and 9.2.5 of [3]). Digital isomorphisms may also be used in pure mathematics, especially in combinatorics.

2. Digital Isomorphisms for Binary Images on a Rectangular Grid

As we explained in the Introduction, a picture digitized on a rectangular grid can be analysed by its Euclidean topological structure or by its digital structure as a set of vertices on which two neighbourhood relations are defined. How are these two structures related? We will answer this question in this section. We will establish a link between the digital structure of the rectangular grid and the topology of the plane. With this correspondence it will be possible to derive from a Euclidean plane homeomorphism ψ between two topological images formed by black and white pixels a relation $\theta = \theta(\psi)$ between the two corresponding digital binary rectangular grid images, which will be called a *digital isomorphism* between them. We will then analyze the main properties of θ, called *totality*, *frame preservation*, *image preservation* and *adjacency preservation*. We will finally generalize our definition and take these four properties as the general definition of a digital isomorphism between binary rectangular grid images, regardless whether it is derived from a Euclidean plane homeomorphism ψ or not.

Let us now describe in a precise way the structure of binary rectangular grid images. Let G be a rectangular grid whose pixels are embedded in the Euclidean plane Π. We can assume that the pixels have size 1 and so G can be identified with a set of ordered pairs (a, b) of integers ($1 \leqslant a \leqslant M$ and $1 \leqslant b \leqslant N$), where M and N are respectively the number of rows and the number of columns of the grid. The structure of G is not determined by open or closed sets, but by two adjacency relations on the pixels, the 4- and 8-adjacencies, which are illustrated in Fig. 2. Note that (a, b) and (c, d) are 4-adjacent if and only if $|a-c| + |b-d| = 1$, while they are 8-adjacent if and only if $\max(|a-c|, |b-d|) = 1$. A *binary image* on G is a map $G \to \{0, 1\}$, associating to each pixel its colour: black $(=1)$ or white $(=0)$.

Now let us recall certain standard definitions of digital image analysis (see, e.g., Chap. 9 of [3]):

DEFINITION 1. Let k be 4 or 8. Two subsets A and B of G are k-adjacent if there is some $a \in A$ and $b \in B$ such that a and b are k-adjacent. A k-path is a chain $x_0, x_1, \ldots, x_n$ such that x_i is k-adjacent to x_{i-1} for $i = 1, \ldots, n$. A subset X of G is k-connected if and only if for every $p, q \in X$, there is a k-path contained in X which joins p to q. For a subset Y of G, the maximal k-connected subsets of Y are called the connected components of Y. Every k-connected subset Z of Y belongs to a unique k-connected component of Y, which is equal to the union of all k-connected subsets of Y having a nonvoid intersection with Z. In particular, the k-connected components of Y form a partition of Y.

Given a binary image $\mathscr{I}$ on G with a set I_1 of black pixels and a set I_0 of white pixels, one chooses always opposite adjacencies on I_1 and I_0 (see Sect. 9.1.1 of [3]). The reason can easily be explained with a simple example. Consider the topological configuration of 4 pixels in the Euclidean plane shown in Fig. 3, where the pixels are in fact surfaces in the plane and let x be the point at the intersection of the 4 pixels. If x is black, then the two black pixels are adjacent, while the two white ones are not; one chooses then the 8-adjacency on I_1 and the 4-adjacency on I_0. On the other hand, if x is white, then the two white pixels are adjacent, while the two black ones are not; one chooses then the 8-adjacency on I_0 and the 4-adjacency on I_1.

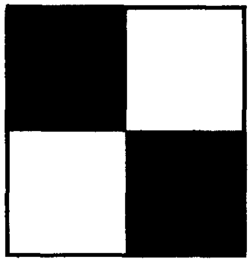


FIGURE 3

This simple example uses a correspondence between the digital structure of G and the topology of the Euclidean plane Π, associating to every pixel in the digital image the square surface which it represents in the topological image. We can pursue in this way and make this correspondence more formal. We will show that the 4-adjacency corresponds to the connectivity of open subsets of the plane, while the 8-adjacency corresponds to the connectivity of closed subsets of the plane. As the complement of an open set is closed and vice versa, this will explain the choice of opposite adjacencies for I_1 and I_0. This same correspondence applied to Euclidean plane homeomorphisms will allow us to construct digital isomorphisms.

Let us recall first some elementary facts of Euclidean plane topology. Let $T \subseteq \Pi$. Then the *border* $\delta(T)$ of T is the set of points of Π at distance 0 from both T and $\Pi \backslash T$. Then T is *closed* if and only if $\delta(T) \subseteq T$ and T is *open* if and only if $T \cap \delta(T) = \emptyset$. The set $T \cup \delta(T)$ is called the *closure* of T, it is the intersection of all closed sets containing T, and we write it $\bar{T}$. The set $T \backslash \delta(T)$ is called the *interior* of T, it is the union of all open sets contained in T, and we write it T°. We say that T is *connected* if and only if there does *not* exist two open subsets O_1 and O_2 of Π such that:

$$
\begin{gathered}
O_1 \cap T \neq \emptyset, \\
O_2 \cap T \neq \emptyset, \\
O_1 \cap O_2 \cap T = \emptyset,
\end{gathered}
\tag{1}
$$

and

$$T \subseteq O_1 \cup O_2.$$

By duality, we have the following: T is connected if and only if there does *not* exist two closed subsets C_1 and C_2 of Π such that:

$$
\begin{gathered}
T \nsubseteq C_1, \\
T \nsubseteq C_2, \\
T \cap C_1 \cap C_2 = \emptyset,
\end{gathered}
\tag{2}
$$

and

$$T \subseteq C_1 \cup C_2.$$

It is easily seen that a union of connected subsets of Π containing a common point x is connected. Thus every subset of Π subdivides into its connected components.

Let us now give the correspondence between connectivity in Π and 4- and 8-connectivity in G. We recall that G is embedded in Π and that every pixel $p \in G$ corresponds to a square surace in Π. Let $S(p)$ be the (topologically) closed square corresponding in the topological image to the pixel p in the digital image. Given $H \subseteq G$, let $S(H)$ be the union of all $S(p)$ for $p \in H$. Then the following holds for any $H \subseteq G$:

PROPOSITION 1. (a) *H is 4-connected if and only if $S(H)^\circ$ is connected. Otherwise, if $H_1, \ldots, H_c$ are the 4-connected components of H, then the connected components of $S(H)^\circ$ are the sets $S(H_i)^\circ = S(H_i) \cap S(H)^\circ$ for $i = 1, \ldots, c$.*

(b) *H is 8-connected if and only if $S(H)$ is connected. Otherwise, if $H^1, \ldots, H^d$ are the 8-connected components of H, then the connected components of $S(H)$ are the sets $S(H^i)$ for $i = 1, \ldots, d$.*

Proof. (a) Let H_i be a 4-connected component of H; then $\delta(S(H_i)) \subseteq \delta(S(H))$. Indeed, let x be a point of $\delta(S(H_i))$, and suppose that $x \in S(p)$ for some $p \in H_i$. If x is a corner point of $S(p)$, then one of the three other pixels y_i ($i = 1, 2, 3$) such that the border of $S(y_i)$ contains x is outside H, because if they were all in H, they would belong to the same 4-connected component of H as p, in other words to H_i, and so x would not be in the border of $S(H_i)$. On the other hand, if x is not a corner point of $S(p)$, then there is a unique pixel q such that $S(p)$ intersects $S(q)$ in a segment containing x, and q is outside H, otherwise it would belong to H_i and x would not be in the border of $S(H_i)$. Thus in any case $x \in \delta(S(z))$ for some $z \notin H$, in other words $x \in \delta(S(H))$.

Therefore $\delta(S(H_i)) = \delta(S(H)) \cap S(H_i)$ and so $S(H_i)^\circ = S(H_i) \setminus \delta(S(H_i)) = S(H_i) \setminus (\delta(S(H)) \cap S(H_i)) = S(H_i) \setminus \delta(S(H)) = S(H_i) \cap S(H)^\circ$. Now, given two 4-adjacent pixels p and q in H_i, the set $S(\{p, q\}) \cap S(H)^\circ$ is connected. As H_i is 4-connected, it follows thus that $S(H_i)^\circ$ is connected. Given another 4-connected component H_j of H, $S(H_j)^\circ$ is also connected. As $S(H_i)^\circ \cap S(H_j)^\circ = \varnothing$, by taking $O_1 = S(H_i)^\circ$, $O_2 = S(H_j)^\circ$ and $T = O_1 \cup O_2$, (1) holds and so T is not connected. Thus $S(H_i)^\circ$ is a connected component of $S(H)^\circ$.

(b) Let H^i be an 8-connected component of H. Given two 8-adjacent pixels p and q in H^i, $S(\{p, q\})$ is connected. Thus $S(H^i)$ must be connected. Given another 8-connected component H^j of H, $S(H^j)$ is also connected. As $S(H^i) \cap S(H^j) = \emptyset$, by taking $C_1 = S(H^i)$, $C^2 = S(H^j)$ and $T = C_1 \cup C_2$, (2) holds and so T is not connected. Thus $S(H^i)$ is a connected component of $S(H)$. ▮

This result leads to a clear explanation of the choice of opposite adjacencies for white and black pixels in binary images. Let $\mathscr{I}$ be a binary image on G with I_0 and I_1 as sets of white and black pixels, respectively. These two sets correspond to the two closed surfaces $S(I_0)$ and $S(I_1)$ in the plane, which intersect in their border. We can make them disjoint in two ways:

(i) We take $S(I_0)$ and $\Pi \backslash S(I_0) = S(I_1)^\circ$. Their connected components correspond to the 8-connected components of I_0 and the 4-connected components of I_1 respectively.

(ii) We take $S(I_1)$ and $\Pi \backslash S(I_1) = S(I_0)^\circ$. Their connected components correspond to the 8-connected components of I_1 and the 4-connected components of I_0 respectively.

Now that we have established a correspondence between the Euclidean plane topology and the digital structure of the rectangular grid, we will see how it can be extended to isomorphisms. From a Euclidean plane homeomorphism ψ we will derive a *binary rectangular grid digital isomorphism* $\theta(\psi)$ which will not be a bijection, nor even a mapping, but a relation between the pixels of two binary rectangular grid images.

We said earlier in the Introduction that if we replace in the two images of Fig. 1 the digital pixels by the topological surfaces to which they correspond, then the two resulting Euclidean plane images are homeomorphic. We will pursue this idea further. Consider two rectangular grids G and G' on which we define the two binary images $\mathscr{I}$ and $\mathscr{I}'$, respectively. We suppose that there is a Euclidean plane homeomorphism ψ such that $\psi(S(I_0)) = S(I'_0)$ and $\psi(S(I_1)) = S(I'_1)$. Of course, ψ does not necessarily map a square $S(p)(p \in G)$ onto a square $S(p')$ $(p' \in G')$, but $\psi(S(p))$ can intersect various squares $S(p')$. This leads to a relation between p and those pixels p' such that $\psi(S(p))$ intersects $S(p')$ and this intersection is not limited to their border. More formally, we make the following:

DEFINITION 2. Given the homeomorphism $\psi: (S(G), \mathscr{I}) \to (S(G'), \mathscr{I}')$, let $\theta = \theta(\psi)$ be the following relation between G and G': for $p \in G$ and $p' \in G'$, $p\theta p'$ if and only if

$$(\psi(S(p)) \cap S(p'))^\circ = \psi(S(p)^\circ) \cap S(p')^\circ \neq \emptyset,$$

or equivalently

$$(S(p) \cap \psi^{-1}(S(p')))^\circ = S(p)^\circ \cap \psi^{-1}(S(p')^\circ) \neq \emptyset. \qquad (3)$$

Then the relation θ is the digital isomorphism between the two digital images $(G, \mathscr{I})$ and $(G', \mathscr{I}')$ derived from the Euclidean plane homeomorphism ψ.

This type of relation derived from a Euclidean plane homeomorphism will be called a *derived* (*binary rectangular grid*) *digital isomorphism.* Now we would like to define a digital isomorphism ϕ for binary rectangular grid images independently from the Euclidean plane homeomorphisms. The way to do it is to investigate the basic properties of the constructed isomorphism θ and infer from them the requirements for a digital isomorphism ϕ in general. For this purpose, let us make first another definition:

DEFINITION 3. To the relation θ between the two digital images $(G, \mathscr{I})$ and $(G', \mathscr{I}')$ we associate the two maps $\rho = \rho_\theta$ and $\lambda = \lambda_\theta$ constructed as follows: for every $X \subseteq G$ and $X' \subseteq G'$ we set:

$$\begin{aligned} X\rho &\doteq \{q' \in G' \mid \exists q \in X \text{ such that } q\theta q'\}, \\ X'\lambda &\doteq \{q \in G \mid \exists q' \in X' \text{ such that } q\theta q'\}, \end{aligned} \tag{4}$$

and for $p \in G$ and $p' \in G'$ we write $p\rho$ and $p'\lambda$ for $\{p\}\,\rho$ and $\{p'\}\,\lambda$.

Thus to the sets $X \subseteq G$ and $X' \subseteq G'$ correspond through θ the sets $X\rho \subseteq G'$ and $X'\lambda \subseteq G$, respectively. Now we can state four important properties of θ, λ and ρ:

(1°) *Totality.* For every $p \in G$ and $p' \in G'$ we have

$$p\rho \neq \varnothing \qquad \text{and} \qquad p'\lambda \neq \varnothing. \tag{5}$$

(2°) *Frame preservation.* In a finite rectangular grid G, the pixels of the first and last rows and columns form what we call the *frame* FG of that grid. These pixels are particular with respect to both the digital structure of G and the Euclidean plane topology. Indeed, if $p \in FG$, then p has less than 8 neighbours in G, and $S(p)$ intersects $\delta(S(G))$. As ψ must map $\delta(S(G))$ onto $\delta(S(G'))$, we obtain the following: for $X \subseteq G$ and $X' \subseteq G'$ we have

$$\text{if } X \cap FG \neq \varnothing, \quad \text{then } X\rho \cap FG' \neq \varnothing$$

and

$$\text{if } X' \cap FG' \neq \varnothing, \quad \text{then } X'\lambda \cap FG \neq \varnothing. \tag{6}$$

An equivalent form is the following: for every $p \in G$ and $p' \in G'$ we have

$$\text{if } p \in FG, \quad \text{then } p\rho \cap FG' \neq \varnothing$$

and

$$\text{if } p' \in FG', \quad \text{then } p\lambda \cap FG \neq \varnothing. \tag{7}$$

For an infinite grid, the frame is the infinity, and the condition (6) must be modified as follows:

If $|G| = \infty$, then one replaces $X \cap FG \neq \varnothing$ and $X'\lambda \cap FG \neq \varnothing$ by $|X| = \infty$ and $|X'\lambda| = \infty$, respectively. (8)

If $|G'| = \infty$, then one replaces $X' \cap FG' \neq \varnothing$ and $X\rho \cap FG' \neq \varnothing$ by $|X'| = \infty$ and $|X\rho| = \infty$, respectively. (9)

Note that one can have a digital isomorphism θ between a finite digital image and an infinite one.

(3°) *Image preservation.* For every $p \in G$ and $p' \in G'$,

$$\text{if } p\theta p', \text{ then } p \text{ and } p' \text{ have the same colour.} \tag{10}$$

(4°) *Adjacency preservation.* It is expected that θ must respect the digital structure of the grid in the same way as ψ respects the Euclidean plane topology. Thanks to the correspondence between the 4- and 8-connectivity in the rectangular grid and the connectivity of open and closed subsets of the plane II, we obtain the following result: for every $k = 4$ or 8, $X \subseteq G$ and $X' \subseteq G'$,

$$\text{If } X \text{ is } k\text{-connected,} \quad \text{then } X\rho \text{ is } k\text{-connected.}$$

$$\text{If } X' \text{ is } k\text{-connected,} \quad \text{then } X'\lambda \text{ is } k\text{-connected.} \tag{11}$$

Let us give a short proof of it: (i) If X is 4-connected, then $S(X)^\circ$ is connected and so $\psi(S(X)^\circ) = \psi(S(X))^\circ$ is connected and intersects $S(p') \cap S(X\rho)^\circ$ for any $p' \in X\rho$; as $\psi(S(X))^\circ \subseteq S(X\rho)^\circ$, this means that $S(X\rho)^\circ$ must be connected, and $X\rho$ is 4-connected. (ii) If X is 8-connected, then $S(X)$ is connected and thus so is $\psi(S(X))$, which intersects $S(p')$ for any $p' \in X\rho$; thus $S(X\rho)$ must be connected and so $X\rho$ is 8-connected. (iii) The proof for X' and $X'\lambda$ is the same.

These are the 4 main properties of the digital isomorphism θ derived from the homeomorphism ψ. They will be the requirements for a relation between $(G, \mathscr{I})$ and $(G', \mathscr{I}')$ to be a digital isomorphism. However we must make the following remark: With the construction of θ from ψ that we have made, the digital isomorphism respects both the 4- and the 8-adjacencies for both black and white pixels. This is due to the fact that for $j = 0, 1$ we have $\psi(S(I_j)) = S(I'_j)$ and $\psi(S(I_j)^\circ) = S(I'_j)^\circ$. If one whishes to consider only k-adjacency on black pixels, k'-adjacency on white ones (where $k = 4$ or 8 and $k' = 12 - k$) and 4-adjacency between black pixels and white ones, then one can consider only the following weaker adjacency preservation property of θ; for $X_0 \subseteq I_0$, $X_1 \subseteq I_1$, $X'_0 \subseteq I'_0$ and $X'_1 \subseteq I'_1$, we have:

$$\begin{array}{ll} \text{If } X_0 \text{ is } k'\text{-connected,} & \text{then } X_0\rho \text{ is } k'\text{-connected.} \\ \text{If } X_1 \text{ is } k\text{-connected,} & \text{then } X_1\rho \text{ is } k\text{-connected.} \\ \text{If } X'_0 \text{ is } k'\text{-connected,} & \text{then } X'_0\lambda \text{ is } k'\text{-connected.} \\ \text{If } X'_1 \text{ is } k\text{-connected,} & \text{then } X'_1\lambda \text{ is } k\text{-connected.} \\ \text{If } X_0 \text{ and } X_1 \text{ are 4-adjacent,} & \text{then } X_0\rho \text{ and } X_1\rho \text{ are 4-adjacent.} \\ \text{If } X'_0 \text{ and } X'_1 \text{ are 4-adjacent,} & \text{then } X'_0\lambda \text{ and } X'_1\lambda \text{ are 4-adjacent.} \end{array} \tag{12}$$

We have thus two different expressions (11) and (12) for the adjacency preservation property, the latter being only a restriction of the former. They will be called *total* and *partial adjacency preservation*, respectively. In the next section we will consider other possible forms to be taken by the adjacency preservation condition for a digital structure.

Let us sum up. We have derived the digital isomorphism θ from the Euclidean plane homeomorphism ψ thanks to (3). We have found that θ has four basic properties:

(1°) Totality (5).

(2°) Frame preservation (6) or (7), with the appropriate change (8) or (9) in the case of an infinite grid.

(3°) Image preservation (10).

(4°) Total adjacency preservation (11). A restricted form of it is the partial adjacency preservation (12).

These four properties will now define a digital isomorphism independently from the Euclidean plane topology:

DEFINITION 4. Given the two binary rectangular grid images $(G, \mathscr{I})$ and $(G', \mathscr{I}')$ and a relation ϕ between G and G', then ϕ will be a digital isomorphism if and only if ϕ (together with the two maps λ_ϕ and ρ_ϕ of Definition 3) satisfies the properties of totality, frame preservation, image preservation and adjacency preservation (total (11) or partial (12)).

We feel that these four properties are sufficient, in the sense that they translate into digital terms all the properties of Euclidean plane homeomorphisms. We make even the following:

CONJECTURE. Suppose that we have two binary rectangular grid images $(G, \mathscr{I})$ and $(G', \mathscr{I}')$ and a relation ϕ between them which satisfies (5), (6), (10), and (11). Then there is a Euclidean plane homeomorphism ξ such that ϕ is derived from ξ by (3).

In the next section, we will define in formal terms digital images; this definition includes many discrete structures, for example n-tone images on a rectangular or hexagonal grid, triangulations of surfaces, and even error-correcting codes. We will then translate in these terms the four requirements for a digital isomorphism. Finally, we will show in Section 4 that these requirements are sufficient to prove that digital isomorphisms preserve the main "topological" properties of digital images (connected components, surrounding relations, etc.).

3. A GENERALIZED DEFINITION OF DIGITAL IMAGES AND OF THE DIGITAL ISOMORPHISM

Now that we have described the digital isomorphism in the case of binary rectangular grid images, we can extend this concept to other types of discrete structures which can be considered as digital images in a broader point of view. Let us give here a few examples of structures which are in some way analogous to the binary rectangular grid images:

(1) The most straightforward generalization is to consider n-tone images on a rectangular grid, where $n \geqslant 3$. Here the requirements are the same as in the binary case, apart from the fact that it is now impossible to consider only the k-adjacency between pixels of certain tones and the k'-adjacency ($k' = 12 - k$) between pixels of the other tones. It is thus necessary to take into account both the 4- and the 8-adjacencies between pixels of whatever tone. This means that a digital isomorphism must have the *total* adjacency preservation property (11), not a *partial* one as in (12).

(2) With images on a *hexagonal grid* (where pixels are regular hexagons), the situation is even simpler, since we have only one adjacency relation. There is thus no distinction between *total* and *partial* adjacency preservation. Again the requirements for a digital isomorphism can be directly derived from the binary rectangular grid case.

(3) One can also study 3-dimensional digital images produced by a cubic tessellation of a real image. The small cubes digitizing the picture are usually called *voxels* [4], and we have two adjacency relations: the 6- and the 26-adjacencies. Two voxels (a, b, c) and (a', b', c') are:

$$\text{6-adjacent if } |a-a'| + |b-b'| + |c-c'| = 1,$$

$$\text{26-adjacent if } \max(|a-a'|, |b-b'|, |c-c'|) = 1.$$

Here the 6-adjacency corresponds to the connectivity of open sets, while the 26-adjacency corresponds to the connectivity of closed sets. As in the two-dimensional case, we can define digital isomorphisms with the *total* adjacency preservation condition, and with the *partial* adjacency preservation condition for binary images. Note that one can define two further adjacency relations, called the 18- and the 18′-adjacencies [4].

We will give later in this section some more examples. It appears thus that one can define a digital isomorphism for various types of images, and this can be done in a general fashion by adapting the 4 requirements of *totality, frame preservation, image preservation* and *adjacency preservation* that we found in the particular case of binary rectangular grid images. For this purpose we must first define these digital images in precise mathematical terms.

Let V be a set, called a *space*, whose elements are called *vertices*; here V corresponds to the grid G of Section 2, and the vertices correspond to the pixels of G. We can define on V one or several *adjacency relations* between its vertices (these relations are in fact sets of ordered pairs (p, q) of vertices of V). The only conditions that we require from them is that these relations are nonreflexive and symmetric (in other words, a vertex p is not adjacent to itself, and if p is adjacent to q, then q is adjacent to p). Suppose that we have defined r adjacency relations on V (e.g., $r = 2$ for the rectangular grid). We can label them $\mathscr{K}_0, \ldots, \mathscr{K}_{r-1}$ and we will say that two vertices p and q are $\mathscr{K}_u$-adjacent if $(p, q) \in \mathscr{K}_u$ (for example in the rectangular grid $\mathscr{K}_0$ corresponds to "4" and $\mathscr{K}_1$ corresponds to "8").

From this definition of adjacency, one can derive such concepts as *$\mathscr{K}_u$-adjacent sets, $\mathscr{K}_u$-paths, $\mathscr{K}_u$-connected sets* and *$\mathscr{K}_u$-connected components* in the same way as in Definition 1 for the rectangular grid.

In some cases, the space V can contain a particular subset called the *frame*, and written FV. This is, for example, the case for finite grids embedded in the Euclidean plane, while a finite grid on, say, a sphere has no frame.

Let us now define images on V. Let $T = \{t_0, \ldots, t_{m-1}\}$ be a set of m integers representing tones (or colours). Then an m-tone image on V is a map $\mathscr{I}: V \to T$, associating to each vertex p its tone $\mathscr{I}(p)$. For example, in binary images, $m = 2$, $t_0 = 0$ and $t_1 = 1$, and a vertex p is black if $\mathscr{I}(p) = 1$ and is white otherwise. We can define for each $j = 0, \ldots, m-1$ the set

$$I_j = \mathscr{I}^{-1}(t_j) = \{p \in V \mid \mathscr{I}(p) = t_j\}$$

and so $\mathscr{I}$ can be considered as a partition of the vertices of V into $I_0, \ldots, I_{m-1}$.

The pair $(V, \mathscr{I})$ is called a *digital image*. Now let us translate in these terms the requirements for a digital isomorphism that we found in Section 2 in the special case of binary images on a rectangular grid.

We assume two digital spaces V and V', on which we define r adjacency relations $\mathscr{K}_0, \dots, \mathscr{K}_{r-1}$ on V and $\mathscr{K}'_0, \dots, \mathscr{K}'_{r-1}$ on V'. Of course, each $\mathscr{K}_u$ corresponds to $\mathscr{K}'_u$. We define moreover the two m-tone images $\mathscr{I}: V \to T$ and $\mathscr{I}': V' \to T$. We wish then to define a digital isomorphism ϕ between $(V, \mathscr{I})$ and $(V', \mathscr{I}')$.

First, ϕ must be a relation between V and V'. We define from it the maps λ and ρ as in (4) (but with V and V' instead of G and G'). Then we have only to translate the four requirements of Section 2:

(1°) *Totality.* As in (5): for every $p \in V$ and $p' \in V'$ we have

$$p\rho \neq \varnothing \qquad \text{and} \qquad p'\lambda \neq \varnothing. \tag{5'}$$

(2°) *Frame preservation.* As we said above, not every space V has a frame. Thus, if V and V' are *finite* and without frame, then this requirement must be omitted. Otherwise we simply take the same condition as (6), or equivalently (7): for $X \subseteq V$ and $X' \subseteq V'$ we have

$$\text{if } X \cap FV \neq \varnothing, \quad \text{then } X\rho \cap FV' \neq \varnothing$$

and

$$\text{if } X' \cap FV' \neq \varnothing, \text{ then } X'\lambda \cap FV \neq \varnothing, \tag{6'}$$

or simply for every $p \in V$ and $p' \in V'$ we have

$$\text{if } p \in FV, \quad \text{then } p\rho \cap FV' \neq \varnothing$$

and

$$\text{if } p' \in FV', \text{ then } p\lambda \cap FV \neq \varnothing, \tag{7'}$$

with the same modification as (8) or (9) when V or V' is infinite:

If $|V| = \infty$, then one replaces $X \cap FV \neq \varnothing$ and $X'\lambda \cap FV \neq \varnothing$ by $|X| = \infty$ and $|X'\lambda| = \infty$, respectively. (8′)

If $|V'| = \infty$, then one replaces $X' \cap FV' \neq \varnothing$ and $X\rho \cap FV' \neq \varnothing$ by $|X'| = \infty$ and $|X\rho| = \infty$, respectively. (9′)

As for the square grid, one can have a digital isomorphism θ between a finite digital image and an infinite one. If V has a frame FV but V' has no frame, then we define $FV' \doteq FV\rho$, and it is easily seen that the frame preservation condition (7′) is satisfied.

(3°) *Image preservation.* As in (10): for every $p \in V$ and $p' \in V'$ we have

$$\text{if } p\theta p', \text{ then } p \text{ and } p' \text{ have the same colour.} \tag{10'}$$

(4°) *Adjacency preservation.* In Section 2, we found two possible conditions (11) and (12), called *total* and *partial* adjacency preservation. As we noted in the first example of this section, there are cases where only the total adjacency preservation condition is to be considered. Now this one can be translated as follows: for every $u = 0, ..., r-1$, $X \subseteq V$ and $X' \subseteq V'$,

$$\begin{aligned} &\text{If } X \text{ is } \mathscr{K}_u\text{-connected,} && \text{then } X\rho \text{ is } \mathscr{K}'_u\text{-connected.} \\ &\text{If } X' \text{ is } \mathscr{K}'_u\text{-connected,} && \text{then } X'\lambda \text{ is } \mathscr{K}_u\text{-connected.} \end{aligned} \tag{13}$$

Now let us translate the partial adjacency preservation condition (12). Given two vertices with respective tones t_i and t_j ($i, j = 0, ..., m-1$), we must take into account only certain adjacencies between them. Let α_{ij} be the set of all $u = 0, ..., r-1$ such that the adjacencies $\mathscr{K}_u$ (or $\mathscr{K}'_u$ for V') are taken into account between vertices having respective tones t_i and t_j. We obtain thus a symmetric matrix (α_{ij}), which will be called the *adjacency matrix.* Now (12) translates as follows: for every $i, j = 0, ..., r-1$, $P_i \subseteq I_i$, $P'_i \subseteq I'_i$, $P_j \subseteq I_j$ and $P'_j \subseteq I'_j$, $u \in \alpha_{ii}$ and $v \in \alpha_{ij}$,

$$\begin{aligned} &\text{If } P_i \text{ is } \mathscr{K}_u\text{-connected,} && \text{then } P_i\rho \text{ is } \mathscr{K}'_u\text{-connected.} \\ &\text{If } P'_i \text{ is } \mathscr{K}'_u\text{-connected,} && \text{then } P'_i\lambda \text{ is } \mathscr{K}_u\text{-connected.} \\ &\text{If } P_i \text{ and } P_j \text{ are } \mathscr{K}_v\text{-adjacent,} && \text{then } P_i\rho \text{ and } P_j\rho \text{ are } \mathscr{K}'_v\text{-adjacent.} \\ &\text{If } P'_i \text{ and } P'_j \text{ are } \mathscr{K}'_v\text{-adjacent,} && \text{then } P'_i\lambda \text{ and } P'_j\lambda \text{ are } \mathscr{K}_v\text{-adjacent.} \end{aligned} \tag{14}$$

It is indeed easily seen that in the case of binary images on a rectangular grid, (12) is nothing but (14) for the adjacency matrix

$$\begin{pmatrix} u' & 0 \\ 0 & u \end{pmatrix}, \tag{15}$$

where $\mathscr{K}_0$, $\mathscr{K}_u$, and $\mathscr{K}_{u'}$ are the 4-, k-, and k'-adjacencies.

On the other hand, the total adjacency preservation corresponds to the adjacency matrix whose entries are all equal to $\{0, ..., r-1\}$.

Conditions (1°)–(4°) together make the definition of the digital isomorphism ϕ between $(V, \mathscr{I})$ and $(V', \mathscr{I}')$. Let us illustrate this definition on some new examples (see also those at the beginning of this section):

(i) Consider the two images of Fig. 1. Then the relation ϕ between V and V' defined by

$$p\phi p' \text{ if and only if } p \text{ and } p' \text{ have the same colour}$$

is a digital isomorphism with *total* adjacency preservation.

(ii) Let S be a bounded closed 2-dimensional surface (in the usual topological sense) in the 3-dimensional Euclidean space. Consider a triangulation $\mathcal{T} = (V, E, F)$ of that surface S with vertices, edges and faces. Such a triangulation is used for the calculation of Euler numbers. Now if we have another triangulation $\mathcal{T}' = (V, E', F')$ of that same surface, it leads to the same value of the Euler number, and so its structure is in some way equivalent to that of $\mathcal{T}$. This equivalence can be expressed in terms of digital images as follows: to a triangulation $\mathcal{T}$ we associate the dual tessellation $\mathcal{T}^* = (V^*, E^*, F^*)$, where to a vertex $v \in V$ corresponds the face $v^* \in F^*$. Now we can consider the vertices $v \in V$ as digital pixels and the corresponding faces $v^* \in F^*$ as the corresponding topological pixels, while the edges in E induce an adjacency relation on V. We proceed then as in Section 2, and we define the relation θ between V and V' by setting for $v \in V$ and $v' \in V'$:

$$v\theta v' \qquad \text{if and only if } (v^* \cap v'^*)^\circ \neq \varnothing. \tag{3'}$$

Then θ will be a digital isomorphism between the two triangulations: it will satisfy the properties of *totality* and *adjacency preservation*, while the requirements of *frame preservation* and *image preservation* are not relevant to this case, since the surface is closed and no image is defined on the triangulation. One might make a conjecture similar to the one of Section 2, namely that two isomorphic digital spaces $\mathcal{T}$ and $\mathcal{T}'$ of this type are in fact triangulations of the same object, and that the isomorphism between them is derived from (3′).

(iii) A relatively similar situation occurs if we take two dual polyhedra $\mathcal{P}$ and $\mathcal{P}^*$ having as sets of vertices, edges and faces V, E, F, and V^*, E^*, F^* respectively. The sets E and E^* induce on V and V^* the adjacency relations $\mathcal{K}$ and $\mathcal{K}^*$, respectively. The duality between $\mathcal{P}$ and $\mathcal{P}^*$ means that there is a bijection π mapping V onto F^*, E onto E^* and F onto V^*, such that π reverses the relation of inclusion between vertices, edges and faces. Now we choose the relation ϕ as follows: for any $v \in V$ and $v^* \in V^*$, $v\phi v^*$ if and only if v is in $\pi^{-1}(v^*)$, or equivalently v^* is in $\pi(v)$. Then it is easily seen that ϕ is a digital isomorphism between V and V^*, where (as in the preceeding example) we take into account only the properties of *totality* and *adjacency preservation.* In fact, ρ (and similarly λ) associates to every vertex the set of vertices along a face, and to a pair of adjacent vertices the set of vertices along two adjacent faces.

(iv) Consider now error-correcting codes. Given a space V (generally a vector space over $\{0, 1\}$) one defines a code $C \subset V$, and one considers $E = V \backslash C$ as being the set of errors. One partitions then E into two sets E_c and E_r consisting respectively of the errors which are corrected to an element of the code and of the errors which are rejected. We have then a correction function $\gamma: E_c \to C$ associating to a correctible error e the code element $\gamma(e)$ to which it is corrected. Then this error-correcting code can be seen as a 3-tone image on the set V endowed with an adjacency relation: the 3 tones (say 0, 1, 2) correspond to the sets C, E_c and E_r, and the adjacency relations consists in the pairs $\{e, \gamma(e)\}$ for $e \in E_c$. Given two such error-correcting codes (V, C, E_C, E_r, γ) and $(V', C', E'_C, E'_r, \gamma')$, one can define a digital isomorphism ϕ between them as a relation between V and V' which satisfies the three requirements of *totality*, *image preservation*, and *adjacency preservation* (indeed, as there is no frame, the *frame preservation* condition falls). It is then easily seen that such a relation ϕ induces a bijection between C and C' and transforms γ into γ'.

Example (ii) is particularly interesting, because it deals with digital images used in classical topology to analyze topological images. This is quite the reverse of what we have been doing in Section 2.

In the next section, we will study the main properties of this new type of isomorphism; they will be very similar to those of the Euclidean plane homeomorphisms.

4. Properties of the Digital Isomorphism

We will analyze here the main properties of our digital isomorphism. We will first deal with connected components (in Subsection 4.1), then with surrounding relations (in Subsection 4.2), and finally with the composition of digital isomorphisms (in Subsection 4.3). The reader will readily note the similarity of these properties with those of the Euclidean plane homeomorphism.

We assume that we have two digital images $(V, \mathscr{I})$ and $(V', \mathscr{I}')$ and a digital isomorphism ϕ from the first one to the second one. Before embarking into the properties of connected components, we will state a few elementary facts:

$$\text{If } X \subseteq Y \subseteq V, \quad \text{then } X\rho \subseteq Y\rho \subseteq V'. \tag{16}$$

$$\text{If } X' \subseteq Y' \subseteq V', \quad \text{then } X'\lambda \subseteq Y'\lambda \subseteq V. \tag{17}$$

$$\text{For any } X \subseteq V, \quad X \subseteq X\rho\lambda \subseteq V. \tag{18}$$

$$\text{For any } X' \subseteq V', X' \subseteq X'\lambda\rho \subseteq V'. \tag{19}$$

Now that these minor details are settled, let us analyze the "topological" properties of ϕ.

4.1. *Connected Components*

We recall the sets $I_j \subseteq V$ and $I'_j \subseteq V'$ consisting of all vertices having tone t_j ($j = 0, \ldots, m-1$). We have then the following:

LEMMA 2. *For every* $j = 0, ..., m-1$, $I_j\rho = I'_j$ and $I'_j\lambda = I_j$.

Proof. By the *image preservation* requirement (3°), we have

$$I_j\rho \subseteq I'_j \tag{20}$$

and

$$I'_j\lambda \subseteq I_j. \tag{21}$$

Applying (17) and (16) to (20) and (21), respectively, we get

$$I_j\rho\lambda \subseteq I'_j\lambda \tag{22}$$

and

$$I'_j\lambda\rho \subseteq I_j\rho. \tag{23}$$

But by (18) and (19) we have

$$I_j \subseteq I_j\rho\lambda \tag{24}$$

and

$$I'_j \subseteq I'_j\lambda\rho. \tag{25}$$

Combining (21) with (22) and (24), and (20) with (23) and (25), we get

$$I_j \subseteq I_j\rho\lambda \subseteq I'_j\lambda \subseteq I_j$$

and

$$I'_j \subseteq I'_j\lambda\rho \subseteq I_j\rho \subseteq I'_j,$$

and so the result holds. ∎

Note. The condition stated in this result is equivalent to the *image preservation* condition of ϕ.

Now we will show that ϕ establishes through λ and ρ a bijection between the connected components of I_j and those of I'_j. Recall the sets α_{ij} of all $u \in \{0, ..., r-1\}$ such that the $\mathscr{K}_u$-adjacency is taken into account between vertices of respective tones t_i and t_j ($i, j = 0, ..., m-1$).

PROPOSITION 3. *Let* $i \in \{0, ..., m-1\}$, $u \in \alpha_{ii}$ *and let* $I_i^1, ..., I_i^t$ *be the* $\mathscr{K}_u$-*connected components of* I_i. *Then* I'_i *has* t $\mathscr{K}'_u$-*connected components* $I_i'^1, ..., I_i'^t$, *where* $I_i^w\rho = I_i'^w$ *and* $I_i'^w\lambda = I_i^w$ *for* $w = 1, ..., t$.

Proof. For every $w = 1, \ldots, t$, let $I_i'^w = I_i^w \rho$. Then each $I_i'^w$ is $\mathscr{K}_u'$-connected by the *adjacency preservation* condition (14), and Lemma 2 implies that I_i' is the union of all $I_i'^w$. Therefore, if $I_i'^w$ is not a $\mathscr{K}_u'$-connected component of I_i', then there is some $v \neq w$ such that $I_i'^v$ is $\mathscr{K}_u'$-adjacent to $I_i'^w$. But then $I_i'^v \lambda \cup I_i'^w \lambda$ is $\mathscr{K}_u$-connected and contains I_i^v and I_i^w, which is a contradiction. Thus the sets $I_i'^w$ are the $\mathscr{K}_u'$-connected components of I_i'.

We know that for every $w = 1, \ldots, t$, $I_i^w \rho = I_i'^w$. As $I_i'^w \lambda$ is $\mathscr{K}_u$-connected and contains I_i^w, $I_i'^w \lambda = I_i^w$. ∎

Note that the *adjacency preservation* condition (14) implies that for i, $j = 0, \ldots, m-1$, $u(i) \in \alpha_{ii}$, $u(j) \in \alpha_{jj}$ and $u \in \alpha_{ij}$, a $\mathscr{K}_{u(i)}$-connected component of I_i is $\mathscr{K}_u$-adjacent to a $\mathscr{K}_{u(j)}$-connected component of I_j if and only if the corresponding $\mathscr{K}_{u(i)}'$-connected component of I_i' is $\mathscr{K}_u'$-adjacent to the corresponding $\mathscr{K}_{u(j)}'$-connected component of I_j'.

This fact has an interesting consequence concerning what is called the *k-adjacency tree* in binary rectangular grid images [2]: this tree has as vertices the k-connected components of I_1 and the k'-connected components of I_0 (where $k = 4$ or 8 and $k' = 12 - k$), and it has as edges the pairs of 4-adjacent connected components of I_0 and I_1. Indeed, it is easily seen that with the digital isomorphism ϕ, ρ induces an isomorphism between the two k-adjacency trees, and λ induces the inverse isomorphism.

We can generalize this to digital images in general. Instead of the numbers k and k', we have the adjacency matrix (α_{ij}). Let us assume that for every $i = 0, \ldots, m-1$, there is some $u(i) \in \alpha_{ii}$ such that for every other $u \in \alpha_{ii}$, the $\mathscr{K}_{u(i)}$-adjacency implies the $\mathscr{K}_u$-adjacency (and similarly for the $\mathscr{K}_u'$- and $\mathscr{K}_{u(i)}'$-adjacencies). This is for example the case in the rectangular grid for the 4-adjacency, which implies the 8-adjacency. Then we can define the *adjacency multigraph* $\mathscr{G}(\alpha)$ of the *image* $(V, \mathscr{I})$ for the *adjacency matrix* (α_{ij}) as follows:

The vertices of $\mathscr{G}(\alpha)$ are the $\mathscr{K}_{u(i)}$-connected components of I_i for $i = 0, \ldots, m-1$.

For every $u \in \alpha_{ij}$, where $i, j = 0, \ldots, m-1$, and $u \neq u(i)$ for $j = i$, a u-edge of $\mathscr{G}(\alpha)$ links a vertex corresponding to a $\mathscr{K}_{u(i)}$-connected component of I_i to every vertex corresponding to a $\mathscr{K}_{u(j)}$-connected component of I_j which is $\mathscr{K}_u$-adjacent to it.

We can similarly define the multigraph $\mathscr{G}'(\alpha)$ for $(V', \mathscr{I}')$. We claim that the digital isomorphism ϕ induces an *isomorphism* between these two multigraphs. Let us define more precisely what this means; it is a bijection $\eta: \mathscr{G}(\alpha) \rightarrow \mathscr{G}'(\alpha)$ such that:

(a) For every vertex X of $\mathscr{G}(\alpha)$, X and $\eta(X)$ have the same tone t_i.

(b) Given two vertices X and Y of $\mathscr{G}(\alpha)$, there is a u-edge between X and Y if and only if there is a u-edge between $\eta(X)$ and $\eta(Y)$.

Then we have the following result:

PROPOSITION 4. *Assume a digital isomorphism* ϕ *between* $(V, \mathscr{I})$ *and* $(V', \mathscr{I}')$ *for the adjacency matrix* (α_{ij}). *Then* ρ *induces an isomorphism* η *from* $\mathscr{G}(\alpha)$ *to* $\mathscr{G}'(\alpha)$, *and* λ *induces the inverse isomorphism* η^{-1}. *Moreover, if* V *and* V' *have a frame* (FV *and* FV', *respectively*) *or if they are infinite, then for every vertex* X *of* $\mathscr{G}(\alpha)$, $X \cap FV \neq \varnothing$ (*or* X *is infinite*) *if and only if* $\eta(X) \cap FV' \neq \varnothing$ (*or* $\eta(X)$ *is infinite*).

The proof is straightforward and is left to the reader. This result has an important meaning. Indeed, one considers generally in the picture processing community that the "topological" structure of a binary rectangular grid image is determined by its k-adjacency tree, in which one further specifies which vertex corresponds to the connected component containing the frame. We can similarly say that the adjacency multigraph $\mathscr{G}(\alpha)$, together with the specification of the vertices which intersect the frame, characterizes the digital structure of the image $(V, \mathscr{I})$ for the adjacency matrix (α_{ij}). Then Proposition 4 means that the digital isomorphism ϕ preserves that digital structure.

The reader will note with interest that Proposition 4 admits a converse. Given two images having isomorphic adjacency multigraphs, if this isomorphism has the *frame preservation* property described in Proposition 4, then there exists a digital isomorphism between these images which induces that isomorphism:

PROPOSITION 5. *Let* $(V, \mathscr{I})$ *and* $(V', \mathscr{I}')$ *be two digital images such that there is an isomorphism* η *mapping* $\mathscr{G}(\alpha)$ *onto* $\mathscr{G}'(\alpha)$. *Assume further that if* V *and* V' *have a frame* (FV *and* FV', *respectively*) *or if they are infinite, then for every vertex* X *of* $\mathscr{G}(\alpha)$, $X \cap FV \neq \varnothing$ (*or* X *is infinite*) *if and only if* $\eta(X) \cap FV' \neq \varnothing$ (*or* $\eta(X)$ *is infinite*). *Define the relation* ϕ *between* V *and* V' *as follows: for every* $v \in V$ *and* $v' \in V'$, $v \phi v'$ *if and only if there exist two vertices* X *and* X' *of* $\mathscr{G}(\alpha)$ *and* $\mathscr{G}'(\alpha)$, *respectively, such that* $v \in X$, $v' \in X'$ *and* $\eta(X) = X'$. *Then* ϕ *is a digital isomorphism from* $(V, \mathscr{I})$ *to* $(V', \mathscr{I}')$ *for the adjacency matrix* (α_{ij}), *and* ρ *and* λ *induce* η *and* η^{-1}, *respectively on the adjacency multigraphs* $\mathscr{G}(\alpha)$ *and* $\mathscr{G}'(\alpha)$.

Proof. Let us check that ϕ satisfies the four requirements for a digital isomorphism. The *totality* property follows from the fact that η is a bijection. The *image preservation* requirement is preserved thanks to the property (a) of η.

Let us now show that ϕ satisfies the *frame preservation* condition. Let Y be a subset of V such that $Y \cap FV \neq \varnothing$ (if V is finite) or Y is infinite (if V is infinite). Then one of the following holds:

(1) Y intersects some vertex X of $\mathscr{G}(\alpha)$ such that $X \cap FV \neq \varnothing$ (if V is

finite) or X is infinite (if V is infinite). Then $T\rho$ contains $\eta(X)$, and $\eta(X) \cap FV' \neq \emptyset$ (if V' is finite) or $\eta(X)$ is infinite (if V' is infinite).

(2) Y intersects an infinite number of vertices X of $\mathscr{G}(\alpha)$, and so $Y\rho$ intersects $\eta(X)$ for all these X. Thus $Y\rho$ is infinite.

Thus, in any case, $Y\rho \cap FV' \neq \emptyset$ (if V' is finite) or $Y\rho$ is infinite (if V' is infinite). We show similarly that for any subset Y' of V' such that $Y' \cap FV' \neq \emptyset$ (if V' is finite) or Y' is infinite (if V' is infinite), then $Y'\lambda \cap FV \neq \emptyset$ (if V is finite) or $Y'\lambda$ is infinite (if V is infinite). Thus the *frame preservation* requirement is satisfied.

Let us finally check the *adjacency preservation* condition:

(1) If $Y \subseteq I_i$ for some $i = 0,..., m-1$ and Y is $\mathscr{K}_u$-connected for some $u \in \alpha_{ii}$, then there is a set $\mathscr{S}$ of vertices of $\mathscr{G}(\alpha)$ such that Y intersects each $X \in \mathscr{S}$ and the u-edges induce a connected graph on $\mathscr{S}$. Then $Y\rho$ is the union of all $\eta(X)$, $X \in \mathscr{S}$, and the u-edges induce also a connected graph on $\eta(\mathscr{S})$ (thanks to the property (b) of η). Therefore $Y\rho$ is $\mathscr{K}'_u$-connected. Thus the first statement of (14) is satisfied. A similar argument can be applied for the second statement of (14).

(2) Let $Y_i \subseteq I_i$ and $Y_j \subseteq I_j$ for two distinct $i, j = 0,..., m-1$, such that Y_i is $\mathscr{K}_u$-adjacent to Y_j for some $u \in \alpha_{ij}$. Then there are two vertices X_i and X_j of $\mathscr{G}(\alpha)$ such that $X_i \subseteq I_i$, $X_j \subseteq I_j$, $X_i \cap Y_i \neq \emptyset$, $X_j \cap Y_j \neq \emptyset$ and X_i is $\mathscr{K}_u$-adjacent to X_j. Then $Y_i\rho$ and $Y_j\rho$ contain $\eta(X_i)$ and $\eta(X_j)$, respectively, and $\eta(X_i)$ and $\eta(X_j)$ are $\mathscr{K}'_u$-adjacent by the property (b) of η. Therefore $Y_i\rho$ and $Y_j\rho$ are $\mathscr{K}'_u$-adjacent and so the third statement of (14) is satisfied. A similar argument can be applied for the fourth statement of (14).

Hence ϕ is a digital isomorphism for the adjacency matrix (α_{ij}). It is straightforward to see that ρ and λ induce η and η^{-1} on $\mathscr{G}(\alpha)$ and $\mathscr{G}'(\alpha)$. ∎

The reader will easily realize that the digital isomorphism between the two images of Fig. 1 that we gave at the end of Section 3 is in fact an application of Proposition 5.

Now that we have dealt with the most fundamental "topological" properties of digital images, let us consider some other topics of lesser importance.

4.2. *Surrounding Relations*

The property of surrounding is defined from the frame FV of V, not from the image $\mathscr{I}$. We will thus assume in this subsection that V and V' have a frame or are infinite. Given some $u \in \{0,..., r-1\}$, $X, Y \subseteq V$ such that $X \cap Y = \emptyset$, then we will say that *Y $\mathscr{K}_u$-surrounds X* if and only if for every $\mathscr{K}_u$-connected $Z \subseteq V$ such that $Z \cap X \neq \emptyset$ and $Z \cap FV \neq \emptyset$ (if V is finite) or Z is infinite (if V is infinite), we must have $Z \cap Y \neq \emptyset$. This translates the intuitive notion that in order to "go out" of X, one must "go through" Y.

If X and Y are connected components or unions of connected components of the sets I_j, then the adjacency multigraph for the adjacency matrix (α_{ij}) (with $\alpha_{ij} = \{0,..., r-1\}$) determines whether $Y\ \mathcal{K}_u$-surrounds X or not. It follows then that ρ and λ preserve surrounding relations between such types of sets.

However, the goal of this subsection is to examine to which extent ϕ preserves surrounding relations between arbitrary sets of vertices. Let us first state a few elementary but nonetheless interesting properties of the surrounding relation:

If $Y\ \mathcal{K}_u$-surrounds X and $W \subseteq X$,
then $Y\ \mathcal{K}_u$-surrounds W. (26)

If $Y\ \mathcal{K}_u$-surrounds X_0 and X_1,
then $Y\ \mathcal{K}_u$-surrounds $X_0 \cup X_1$. (27)

Every $Y \subseteq V$ contains a maximal $\mathcal{K}_u$-surrounded set $\mathcal{S}_u(Y)$,
which is $\mathcal{K}_u$-surrounded by Y and contains every $X \subseteq V$
such that $Y\ \mathcal{K}_u$-surrounds X. (28)

If V is finite and $Y\ \mathcal{K}_u$-surrounds X,
then X does not $\mathcal{K}_u$-surround Y. (29)

If $Z\ \mathcal{K}_u$-surrounds Y, $Y\ \mathcal{K}_u$-surrounds X and $Z \cap X \neq \varnothing$,
then $Z\ \mathcal{K}_u$-surrounds X. (30)

The properties (29) and (30) mean that the relation of $\mathcal{K}_u$-surrounding is a strict partial order relation on any family of disjoint subsets (in other words it is nonreflexive, nonsymmetric, and transitive). Note that (29) does not hold for an infinite V. For example, if we take the infinite rectangular grid, which is equivalent to the set of ordered pairs of integers, then we can take

$$X = \{(a, b) \mid \max(|a|, |b|) \text{ is odd}\}$$

and

$$Y = \{(a, b) \mid \max(|a|, |b|) \text{ is even}\},$$

and then X and Y k-surround each other for both $k = 4$ and $k = 8$ (see Fig. 4).

Let us now see to which extent the digital isomorphism ϕ preserves surrounding relations. As the property of $\mathcal{K}_u$-surrounding is based on the $\mathcal{K}_u$-adjacency irrespectively of the tone of the vertices, if we want to preserve it by ϕ, then we must assume that $u \in \alpha_{ij}$ for every $i, j \in \{0,...,$

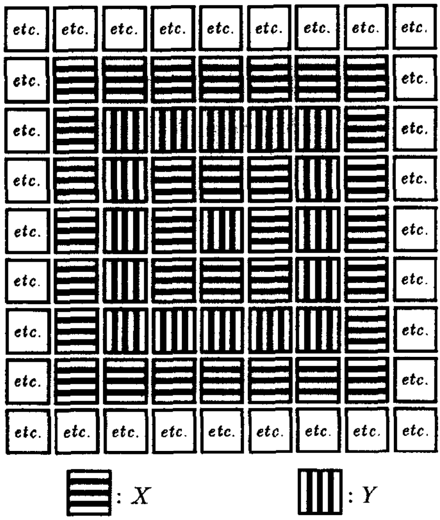


FIGURE 4

$m-1\}$ (this happens for example with the *total* adjacency preservation). We get then the following result:

PROPOSITION 6. *Assume that* $u \in \alpha_{ij}$ *for every* $i, j \in \{0, ..., m-1\}$. *Let* A, $B \subseteq V$ *and* A', $B' \subseteq V'$. *Then*:

(a) *If* B $\mathscr{K}_u$*-surrounds* $A \backslash B$, *then* $B\rho$ $\mathscr{K}'_u$*-surrounds* $A\rho \backslash B\rho$.

(b) *If* $B' \mathscr{K}'_u$*-surrounds* $A' \backslash B'$, *then* $B'\lambda$ $\mathscr{K}_u$*-surrounds* $A'\lambda \backslash B'\lambda$.

Proof. It is sufficient to prove (a), because (b) is proved in the same way if we interchange λ and ρ. Let us first note that

$$B\rho \cap (A\rho \backslash B\rho) = \varnothing. \tag{31}$$

Consider now a subset C' of V' such that C' is $\mathscr{K}'_u$-connected, $C' \cap (A\rho \backslash B\rho) \neq \varnothing$ and $C' \cap FV' \neq \varnothing$ (if V' is finite) or C' is infinite (if V' is infinte). Then we must show that $C' \cap B\rho \neq \varnothing$. Let us note the following three facts:

$$\text{As } C' \cap A\rho \neq \varnothing, \quad C'\lambda \cap A \neq \varnothing \tag{32}$$

by the definition of λ and ρ.

$$\begin{aligned} &\text{As } C' \cap FV' \neq \varnothing \text{ (if } V' \text{ is infinite),} \\ &\quad \text{or } C' \text{ is infinite (if } V' \text{ is infinite),} \\ &\quad C'\lambda \cap FV \neq \varnothing \text{ (if } V \text{ is finite),} \\ &\quad \text{or } C'\lambda \text{ is infinite (if } V \text{ is infinite)} \end{aligned} \tag{33}$$

by the *frame preservation condition.*

$$C'\lambda \text{ is } \mathscr{K}_u\text{-connected} \tag{34}$$

by the *adjacency preservation* condition. As B $\mathscr{K}_u$-surrounds $A \setminus B$, (32), (33) and (34) imply that $C'\lambda \cap B \neq \varnothing$, and so $C' \cap B\rho \neq \varnothing$. This fact, together with (31), means that $B\rho$ $\mathscr{K}'_u$-surrounds $A\rho \setminus B\rho$. ∎

Proposition 6 has the following consequence:

COROLLARY 7. *Let u, A, B, A' and B' be as in Proposition* 6. *Then*:

(i) *If B $\mathscr{K}_u$-surrounds $A'\lambda$, then $B\rho$ $\mathscr{K}'_u$-surrounds A'.*

(ii) *If B' $\mathscr{K}'_u$-surrounds $A\rho$, then $B'\lambda$ $\mathscr{K}_u$-surrounds A.*

Proof. Again we must prove only (i). As B $\mathscr{K}_u$-surrounds $A'\lambda$, $B \cap A'\lambda = \varnothing$. But then $B\rho \cap A' = \varnothing$. Now $A' \subseteq A'\lambda\rho$ by (19). Thus $A' \subseteq A'\lambda\rho \setminus B\rho$. But Proposition 6(i) implies that $B\rho$ $\mathscr{K}'_u$-surrounds $A'\lambda\rho \setminus B\rho$, and so it $\mathscr{K}'_u$-surrounds A' by (26). ∎

Note that the converse of Corollary 7 is false. Take, for example, $V = V' = G$, where G is a rectangular grid; define ϕ by $p\phi q$ if and only if q is an 8-neighbour of p. Then ϕ is a digital isomorphism satisfying the total adjacency preservation condition (11) and with $\lambda = \rho$. Now for every pixel $x \in G$, $x\rho$ 8-surrounds x, but x does not 8-surround $x\lambda = x\rho$.

We will define a particular class of subsets of V (and V') for which Corollary 7 admits a converse. It will contain among others the connected components of the sets I_i. We set

$$\mathscr{X} = \{X \subseteq V \mid X\rho\lambda = X\} \tag{35}$$

and

$$\mathscr{X}' = \{X' \subseteq V' \mid X'\lambda\rho = X'\}. \tag{36}$$

The restriction $\rho_{|\mathscr{X}}$ of ρ to $\mathscr{X}$ and $\lambda_{|\mathscr{X}'}$ of λ to $\mathscr{X}'$ form a bijection $\mathscr{X} \to \mathscr{X}'$ and its inverse.

Now $\mathscr{X}$ and $\mathscr{X}'$ have two particular subsets. Let $\mathscr{C}$ be the set of $X \subseteq V$ such that X is a $\mathscr{K}_u$-connected component of I_i for some $i = 0, \ldots, m-1$ and $u \in \alpha_{ii}$, and let $\mathscr{D}$ be the set of unions of elements of $\mathscr{C}$. Define similarly $\mathscr{C}'$ and $\mathscr{D}'$ for V'. Then $\mathscr{C}$, $\mathscr{D} \subseteq \mathscr{X}$, $\mathscr{C}'$, $\mathscr{D}' \subseteq \mathscr{X}'$, and ρ and λ interchange on the one hand $\mathscr{C}$ and $\mathscr{C}'$, and on the other hand $\mathscr{D}$ and $\mathscr{D}'$. The following result holds for $\mathscr{X}$:

PROPOSITION 8. *Let u be as above. Let X, $Y \in \mathscr{X}$. Then Y $\mathscr{K}_u$-surrounds X if and only if $Y\rho$ $\mathscr{K}'_u$-surrounds $X\rho$.*

This result follows from Corollary 7 by setting $B = Y$, $A = X$, $B' = Y\rho$ and $A' = X\rho$. In particular, if we take $X, Y \in \mathscr{C}$, then we find again what we stated in the beginning of this subsection as a consequence of the preservation by ϕ of the adjacency multigraph.

4.3. *The Composition of Digital Isomorphisms*

It is well known that in ordinary topology, homeomorphisms admit the laws of composition and of inversion. Let us see what happens with digital isomorphisms for this matter.

As we will be dealing with several isomorphisms at the same time, we will write ρ_ϕ and λ_ϕ for the two maps ρ and λ derived from ϕ.

Let ϕ and ϕ' be two digital isomorphisms from $(V, \mathscr{I})$ to $(V', \mathscr{I}')$ and from $(V', \mathscr{I}')$ to $(V'', \mathscr{I}'')$, respectively, for the same adjacency matrix (α_{ij}). Then we define the inverse ϕ^{-1} of ϕ and the composition $\phi \cdot \phi'$ of ϕ by ϕ' as follows: for any $p \in V$, $p' \in V'$ and $p'' \in V''$,

$$p'\phi^{-1}p \text{ if and only if } p\phi p'. \tag{37}$$

$$p(\phi \cdot \phi')\, p'' \text{ if and only if there exists } p' \in V' \text{ such that } p\phi p' \text{ and } p'\phi' p''. \tag{38}$$

Then it is easy to verify that ϕ^{-1} and $\phi \cdot \phi'$ are two digital isomorphisms from $(V', \mathscr{I}')$ to $(V, \mathscr{I})$ and from $(V, \mathscr{I})$ to $(V'', \mathscr{I}'')$, respectively, for the adjacency matrix (α_{ij}). Moreover we have

$$\rho_{\phi^{-1}} = \lambda_\phi$$

and

$$\lambda_{\phi^{-1}} = \rho_\phi; \tag{39}$$

$$\rho_{\phi \cdot \phi'} = \rho_\phi \cdot \rho_{\phi'}$$

and

$$\lambda_{\phi \cdot \phi'} = \lambda_{\phi'} \cdot \lambda_\phi. \tag{40}$$

The composition of digital isomorphisms is associative and we have

$$(\phi \cdot \phi')^{-1} = \phi'^{-1} \cdot \phi^{-1}. \tag{41}$$

The identity relation 1_V on V is a digital isomorphism from (V, I) to itself for any adjacency matrix. However, given a digital isomorphism ϕ, the isomorphisms $\phi \cdot \phi^{-1}$ and $\phi^{-1} \cdot \phi$ are not the identity, but they contain it. In particular we cannot form a group with the digital isomorphisms of $(V, \mathscr{I})$, as it is the case for the homeomorphisms in usual topology.

Let us finally give an interesting consequence of the existence of the identity isomorphism, the inverse ϕ^{-1} and of the composition $\phi \cdot \phi'$. Let us say that a digital image $(V, \mathscr{I})$ *is isomorphic to* a second one $(V', \mathscr{I}')$ if there is a digital isomorphism ϕ from $(V, \mathscr{I})$ to $(V', \mathscr{I}')$; then this relation "is isomorphic to" is an equivalence relation (in other words, it is reflexive, symmetric, and transitive).

5. Conclusion

We have defined a digital isomorphism between digital images as a relation satisfying the following requirements:

(a) totality,

(b) frame preservation,

(c) image preservation,

(d) adjacency preservation (total or partial).

As the last requirement depends on the adjacency matrix (α_{ij}), the digital isomorphism can be chosen to suit the types of adjacency relations that are used in a particular image. For example on binary rectangular grid images we can take into account the k-adjacency between black pixels and the k'-adjacency between white pixels, where $k = 4$ or 8 and $k' = 12 - k$.

The analogy with the Euclidean plane homeomorphisms exhibited in Section 2 in the case of binary rectangular grid images, and the properties proven in Section 4 are convincing arguments in the justification of our definition of the digital isomorphism. If one could prove the conjecture that we make in Section 2 (that every digital isomorphism between binary rectangular grid images with *total* adjacency preservation can be derived from a Euclidean plane homeomorphism), then this would provide a definitive justification of that definition.

Digital isomorphisms can be used to check the validity of various operations which are claimed to preserve the "topology" of digital images, for example, thinning, shrinking, expansion. They can be applied not only to two-dimensional rectangular grid images, but to various types of discrete structures.

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